An Analysis of Asymmetric Rolling Bodies with Nonlinear Aerodynamics

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A nonlinear analysis is performed for the motion of a rolling re-entry vehicle for the case of constant roll rate and for the case of variable roll rate, dynamic pressure, and stability derivatives. Each of the restoring, damping, lag, Magnus, and induced moments is assumed to be a cubic function of the angle-of-attack with variable coefficients. In the constant roll rate case, the derivative expansion version of the method of multiple scales is used to derive equations that characterize the time variation of the amplitudes and phases of the nutation and precession modes. In the variable roll rate case, the generalized version of the method of multiple scales is used to derive equations for the roll rate and the amplitudes and phases of the oscillatory modes of the angle of attack near and away from roll resonance. The roll degree of freedom is coupled with the pitch and yaw degrees of freedom instead of assuming a priori the variation of the roll rate. The analytical solutions are found to be in good agreement with the numerical solutions of the full problem.

Nomenclature

A_n	= complex amplitude functions: $a_n e^{i\theta_n}$, $n = 1, 2, 3$
a_n	= real amplitude function, Eq. (19)
I	= transverse moment of inertia
I_x	= longitudinal moment of inertia
K	= magnitude of trim moment, Eq. (9)
k	= $K/\varepsilon^2 = 0(\omega_0{}^2)$
p	= roll rate
p_0, p_2	= Eq. (43)
q, r	= angular velocities of coordinate system about the y- and z-axes
T_2	= slow time scale: $\varepsilon^2 t$
t	= time
u, v, w	= velocities in x, y, and z directions
x	= longitudinal coordinate
y, z	= transverse coordinates
α	= w/u, Eq. (4)
$\tilde{\alpha}$	= Eq. (7)
β	= v/u, Eq. (4)
α_0, β_0	= values of α and β at $t = 0$
Δ	= $(\omega_1 - \omega_2)^{-1}$
γ	= cubic restoring moment parameter, Eq. (9)
ε	= small parameter proportional to $\lvert\xi\rvert$
ζ	= $q + ir$
η_1	= fast time scale, $\dot{\eta}_1 = \omega_1$
η_2	= fast time scale, $\dot{\eta}_2 = \omega_2$
θ_n	= phase of oscillation modes
λ_{ij}	= Eqs. (20) and (56)
μ_1, μ_2	= linear and cubic damping and lag moment parameter, Eq. (9)
ν_0	= pure rolling moment parameter, Eq. (9)
ν_1	= c.g. offset rolling moment parameter, Eq. (9)
ν_2	= damping rolling moment parameter, Eq. (9)
ξ	= $(v + iw)/u$, Eq. (4)
ξ_1, ξ_3	= Eq. (13)
τ	= dimensionless time, $t\omega_0/2\pi$
ϕ	= roll orientation angle, $\phi = \int p\,dt$
ϕ_0	= orientation of trim moment
χ_1, χ_2	= linear and cubic Magnus and induced moment parameter, Eq. (9)
ω_0	= critical frequency, $(-M_{\alpha 0}/I)^{1/2}$
ω_1	= nutation frequency, Eq. (11)
ω_1'	= $d\omega_1/dT_2$
ω_2	= precession frequency, Eq. (11)
ω_2'	= $d\omega_2/dT_2$

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1. Introduction

TO describe the motion of rolling re-entry bodies at large angles of attack or near roll resonance, the damping as well as the nonlinear aerodynamic stability derivatives must be taken into account. Roll resonance is a condition where the nutation frequency (one of the natural frequencies) is equal to the roll rate (excitation frequency of a body-fixed trim). Since the frequencies of the modes of oscillation change with altitude, the variation of dynamic pressure and stability derivatives with time must also be taken into account. Moreover, the time variation of the roll rate must be taken into account, especially for the roll resonance case.

Murphy[1] investigated the motion of asymmetric bodies using a quasi-linear technique for the constant coefficient, constant roll rate case. He determined equations for the variation of the frequencies and damping rates with amplitude in the case of a cubic static moment, however, his analysis of roll resonance is questionable because he separated the response of the nutational and rolling modes in this case. We will show below that the nutational mode is entrained by the rolling mode and cannot be decoupled. Clare[2] used the method of averaging[3] to determine equations for the amplitudes and the phases for the roll resonance case assuming constant dynamic pressure and stability derivatives. He investigated the conditions for steady-state solutions (perfect resonance) but did not investigate their stability.

One of the purposes of this paper is to extend the work of Murphy and Clare and to analyze the motion of rolling asymmetric bodies with nonlinear aerodynamic moments for the case of constant stability derivatives and constant roll rate using the method of multiple scales[4] (Sec. 3).

Murphy[5] analyzed the effect of a variable roll rate on the roll resonance problem for the case of constant dynamic pressure and linear aerodynamics. The roll mode was decoupled from the pitch and yaw modes so that the effect of the latter modes on the roll mode (such as exists if there is a c.g. offset asymmetry) cannot be ascertained from such an analysis.

A second purpose of this paper is to analyze the motion of rolling re-entry bodies for the case of nonlinear aerodynamics, along with variable roll rate, dynamic pressure and aerodynamic stability derivatives (Secs. 4 and 5). The analysis couples the roll degree of freedom with the pitch and yaw degrees of freedom. The analysis is carried out using the generalized version of the method of multiple scales.[4] Nayfeh[6] used this technique to analyze the case of linear aerodynamics but included variable roll rate, dynamic pressure, and aerodynamic derivatives.

2. Problem Formulation

We introduce a nonrolling, right-handed Cartesian system where the x-axis is the longitudinal axis and the y- and z-axis are the transverse axes as shown in Fig. 1. The translational velocity

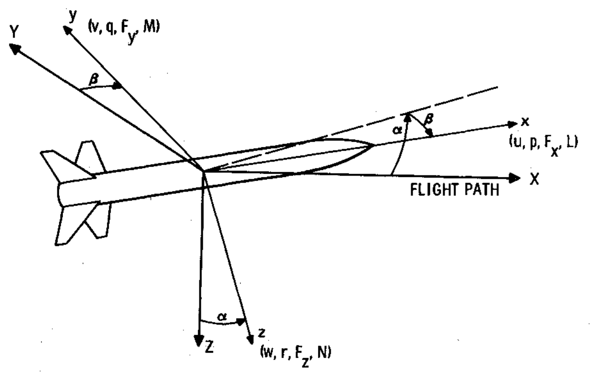

Fig. 1 Coordinate system (X, Y, Z) inertial system, (x, y, z) body fixed, nonrolling system.

components along the x-, y-, and z-axes are given by u, v, and w, respectively, while the coordinate system is rotating with angular velocities q and r about the y- and z-directions and the body is rolling with angular velocity p about its longitudinal axis (i.e., x-axis). We assume that body forces are negligible and that the body has a c.g. offset and a trim angle of attack, but otherwise has mass and aerodynamic symmetry about its longitudinal axis. With these assumptions, the translation equations of motion are

$$F_y = m(\dot v + ru), \quad F_z = m(\dot w - qu) \tag{1}$$

where m is the body mass, and F_y and F_z are the normal forces in the y- and z-directions, respectively. The rotational equations of motion are

$$M = I\dot q + prI_x, \quad N = I\dot r - pqI_x \tag{2}$$

and

$$L = I_x \dot p \tag{3}$$

where I and I_x are the transverse and longitudinal moments of inertia, and L, M, and N are the moments about the x-, y-, and z-axes, respectively.

A complex angle of attack and a complex angular velocity can be defined as

$$\xi = \beta + i\alpha = (v+iw)/u \quad \text{and} \quad \zeta = q + ir \tag{4}$$

respectively, where $|\xi|$ is the tangent of the total angle of attack of the body. We assume that the transverse aerodynamic forces are proportional to the total angle of attack; that is

$$F_y + iF_z = F_0\,\xi \tag{5}$$

Furthermore, the transverse moments and the roll moment are assumed to be‡

$$M + iN = (M_0+iN_0)e^{\phi} - i(M_{\alpha 0}+M_{\alpha 2}\,\xi\bar\xi)\xi + (M_{q0}+M_{q2}\,\xi\bar\xi)\zeta - i(M_{\dot\alpha 0}+M_{\dot\alpha 2}\,\xi\bar\xi)\dot\xi + [(pM_{p\beta}+M_{\beta\phi})_0 + (pM_{p\beta}+M_{\beta\phi})_2\,\xi\bar\xi]\xi \tag{6}$$

$$L = L_0 + L_p p + L_{\tilde\alpha}\tilde\alpha; \quad \tilde\alpha = \text{Imag}\{\xi e^{-i\phi}\} \tag{7}$$

‡ $\dot{}$ denotes differentiation with respect to time, $\bar{}$ denotes complex conjugate.

Here M_0 and N_0 are the trim moments and M_α, M_q, $M_{\dot\alpha}$, $M_{p\beta}$, and $M_{\beta\phi}$ are the restoring, damping, lag, Magnus, and induced moment stability derivatives, respectively. The subscripts 0 and 2 represent the linear and cubic stability derivatives and the usual symmetry of aerodynamic moments has been used. The roll angle is ϕ while L_0, $L_p p$, and $L_{\tilde\alpha}\tilde\alpha$ are the pure, damping, and c.g. offset rolling moments.

Equations (1–7) can be combined to give

$$\ddot\xi - iI_x I^{-1} p\dot\xi + \omega_0^2\xi = \varepsilon K \exp i(\phi+\phi_0) + \gamma|\xi|^2\xi + \varepsilon^2\mu_1\dot\xi + \mu_2|\xi|^2\dot\xi + i\varepsilon^2\chi_1\xi + i\chi_2|\xi|^2\xi \tag{8a}$$

$$\dot\phi = p \tag{8b}$$

$$\dot p = \varepsilon^2\nu_0 + \varepsilon\nu_1\tilde\alpha + \varepsilon^2\nu_2 p, \quad \tilde\alpha = \text{Imag}\{\xi e^{-i\phi}\} \tag{8c}$$

where

$$\begin{aligned}
\omega_0^2 &= -\left[\frac{M_\alpha}{I} + \frac{M_q}{I}\frac{1}{u}\left(\dot u - \frac{F_0}{m}\right)\right]_0, \\
\gamma &= \left[\frac{M_\alpha}{I} + \frac{M_q}{I}\frac{1}{u}\left(\dot u - \frac{F_0}{m}\right)\right]_2, \\
\varepsilon^2\mu_1 &= \frac{(M_{\dot\alpha}+M_q)_0}{I} - \frac{1}{u}\left(\dot u - \frac{F_0}{m}\right), \quad \mu_2 = (M_{\dot\alpha}+M_q)_2 I^{-1} \\
\varepsilon^2\chi_0 &= \frac{(pM_{p\beta}+M_{\beta\phi})_0}{I} - \frac{\ddot u}{u} - \frac{1}{u}\left(\dot u - \frac{F_0}{m}\right)\left(\frac{\dot u}{u} + i\frac{I_x}{I}p\right) + \frac{\dot F_0}{um} \\
\chi_2 &= (pM_{p\beta}+M_{\beta\phi})_2 I^{-1} \\
\varepsilon^2\nu_0 &= L_0/I_x, \quad \varepsilon\nu_1 = L_{\tilde\alpha}/I_x, \quad \varepsilon^2\nu_2 = L_p/I_x \\
\varepsilon K \exp(i\phi_0) &= i(M_0+iN_0)/I \quad \text{with real } K \text{ and } \phi_0
\end{aligned} \tag{9}$$

Here, ε is a small but finite dimensionless quantity of the order of the magnitude of the angle of attack, which has been introduced in the asymmetry, linear damping, and Magnus terms so that μ_1, μ_2 and ν_2 are $0(\omega_0)$ or less, and K, χ_1, χ_2, ν_0, and ν_1 are $0(\omega_0^2)$ or less. Although the symmetry condition holds for small angles of attack, it may not hold for large angles of attack as partially confirmed by experiments on spinning and coning motions of slender cones at supersonic speeds.[7,8] If this is the case, the form of Eq. (8) still holds if μ_2 and χ_2 are related properly to the damping, Magnus, lag, and induced moments. Our analysis is valid in both cases as long as ν_1, χ_1, μ_1, γ, and ω_0 are slowly varying functions of time.

In the absence of damping and nonlinear terms, the solution of Eq. (8) for constant p and ω_0 is given by

$$\xi = A_1 \exp i\omega_1 t + A_2 \exp i\omega_2 t + \varepsilon K(\omega_1 - p)^{-1}(p-\omega_2)^{-1} \times \exp i(\phi+\phi_0) \tag{10}$$

where A_1 and A_2 are complex constants, and

$$\omega_{1,2} = (pI_x/2I) \pm [(pI_x/2I)^2 + \omega_0^2]^{1/2} \tag{11}$$

The frequencies ω_1 and ω_2 are called the nutation and precession frequencies.§ For statically stable bodies (i.e., $\omega_0^2 > 0$) and positive p, ω_1 is positive while ω_2 is negative. Two cases arise depending on whether p is near ω_1 or not. The first case is called roll resonance and the forced response tends to infinity at $p = \omega_1$. Before p approaches ω_1, however, the damping as well as the nonlinear aerodynamic forces will significantly modify the response.

Another source of resonance will appear if one attempts to determine an approximate solution to Eqs. (8) away from roll resonance by iterating on the solution (10). The resultant solution will contain secular terms which make the solution invalid for long times. A solution technique such as quasilinearization or multiple time scales must be used to get valid solutions. The solutions obtained by these methods display what is known as subharmonic resonances which in this case can occur when $p \approx 3\omega_1$. Higher resonances are also possible. We have used the method of multiple scales to analyze 1) the nonlinear resonances of a constant roll rate body in the following section, 2) the nonlinear resonances of a variable roll rate asymmetric body away from roll resonance in Sec. 4, and 3) near roll resonance in Sec. 5.

3. Case of Constant Roll Rate

In this section we assume that the roll rate p, as well as the aerodynamic coefficients, are constant. Although this assumption restricts the problem to high altitudes or wind-tunnel modeling, it will provide a basis for analyzing the more difficult problem of variable roll rate. To determine an approximate solution to Eq. (8a) with $\phi = pt$ using the method of multiple scales, we introduce a slow time scale $T_2 = \varepsilon^2 t$ in addition to the fast scale $T_0 = t$. As mentioned, ε is a small quantity of the order of the magnitude of the angle of attack. With these variables, the time derivative is transformed according to

$$d/dt = (\partial/\partial T_0) + \varepsilon^2(\partial/\partial T_2) \tag{12}$$

and Eq. (8a) is transformed into a partial differential equation. We assume that ξ possesses a uniformly valid expansion of the form

$$\xi(t) = \varepsilon\xi_1(T_0, T_2) + \varepsilon^3\xi_3(T_0, T_2) + \dots \tag{13}$$

The condition that Eq. (13) is uniformly valid for all t is equivalent to the condition that

$$\xi_3/\xi_1 < \infty \quad \text{for all} \quad T_0 \quad \text{and} \quad T_2 \tag{14}$$

Thus, uniformly valid solutions are obtained by determining the functional dependence of ξ_1 and ξ_3 on T_2 in order to satisfy Eq. (14).

Substituting Eqs. (12 and 13) into Eq. (8a), noting that $\mu_1 = 0(\omega_0)$ and $\chi_1 = 0(\omega_0{}^2)$, and equating coefficients of like powers of ε, we get

$$\frac{\partial^2\xi_1}{\partial T_0{}^2} - ip\frac{I_x}{I}\frac{\partial\xi_1}{\partial T_0} + \omega_0{}^2\xi_1 = K\exp i(pT_0 + \phi_0) \tag{15}$$

$$\frac{\partial^2\xi_3}{\partial T_0{}^2} - ip\frac{I_x}{I}\frac{\partial\xi_3}{\partial T_0} + \omega_0{}^2\xi_3 = -2\frac{\partial^2\xi_1}{\partial T_0\,\partial T_2} + ip\frac{I_x}{I}\frac{\partial\xi_1}{\partial T_2} + (\gamma + i\chi_2)|\xi_1|^2\xi_1 + \mu_1\frac{\partial\xi_1}{\partial T_0} + \mu_2|\xi_1|^2\frac{\partial\xi_1}{\partial T_0} + i\chi_1\xi_1 \tag{16}$$

The general solution of (15) is

$$\xi_1 = A_1(T_2)\exp(i\omega_1 T_0) + A_2(T_2)\exp(i\omega_2 T_0) + A_3\exp(ipT_0) \tag{17}$$

where

$$A_3 = a_3\exp(i\phi_0) = K(\omega_1 - p)^{-1}(p - \omega_2)^{-1}\exp(i\phi_0)$$

where the nutation and precession frequencies ω_1 and ω_2 are defined in Eq. (11). Substituting Eq. (17) into Eq. (16) gives

$$\frac{\partial^2\xi_3}{\partial T_0{}^2} - ip\frac{I_x}{I}\frac{\partial\xi_3}{\partial T_0} + \omega_0{}^2\xi_3 = Q_1\exp(i\omega_1 T_0) + Q_2\exp(i\omega_2 T_0) + Q_3\exp(ipT_0) + \sum_{n=1}^{2}\{m_n A_n{}^2\bar{A}_s\exp[i(2\omega_n - \omega_s)T_0] + m_n A_n A_s \bar{A}_3\exp[i(\omega_n + \omega_s - p)T_0] + m_n A_n{}^2\bar{A}_3\exp[i(2\omega_n - p)T_0] + m_3\bar{A}_n A_3{}^2\exp[i(2p - \omega_n)T_0] + (m_n + m_3)A_n\bar{A}_s A_3\exp[i(p - \omega_s + \omega_n)T_0]\} \tag{18}$$

where the m's and Q's are defined in Appendix A and $s = 2$ when $n = 1$ while $s = 1$ when $n = 2$. The particular solution of Eq. (18) contains secular terms of the form $T_0\exp(i\omega_1 T_0)$ and $T_0\exp(i\omega_2 T_0)$ which make ξ_3/ξ_1 unbounded as $T_0 \to \infty$. To get a uniformly valid expansion, A_1 and A_2 are chosen so that the secular terms vanish; that is, $Q_1 = Q_2 \equiv 0$. Letting $A_n = a_n\exp i\theta_n$ with real a_n and θ_n, separating real and imaginary parts in the expressions for Q_1 and Q_2, we get[9]

$$da_n/dT_2 = \lambda_{1n}a_n + \lambda_{2n}a_n{}^3 + \lambda_{3n}a_n a_s{}^2 + \lambda_{4n}a_n a_3{}^2 \tag{19a}$$

$$d\theta_n/dT_2 = (-1)^n\gamma\Delta(a_n{}^2 + 2a_s{}^2 + 2a_3{}^2) \tag{19b}$$

where $n = 1, 2$, $s = 2, 1$, and

$$\lambda_{1n} = (-1)^{n+1}\Delta(\omega_n\mu_1 + \chi_1) \tag{20a}$$

$$\lambda_{2n} = (-1)^{n+1}\Delta(\omega_n\mu_2 + \chi_2) \tag{20b}$$

$$\lambda_{3n} = (-1)^{n+1}\Delta[(\omega_1 + \omega_2)\mu_2 + 2\chi_2] \tag{20c}$$

$$\lambda_{4n} = (-1)^{n+1}\Delta[(\omega_n + p)\mu_2 + 2\chi_2] \tag{20d}$$

$$\Delta = (\omega_1 - \omega_2)^{-1} \tag{20e}$$

Equations (19–20) show that the nonlinear part of the pitching moment characterized by γ does not affect the amplitudes a_1 and a_2 in this case. However, it affects the phases, thereby decreasing the magnitudes of the nutation and precession frequencies. The asymmetry also decreases these frequencies.

For a positive p, ω_1 is positive while ω_2 is negative. In this case, Eq. (19) shows that μ_1 is stabilizing (tends to dampen amplitudes) or destabilizing (tends to increase amplitudes) depending on whether it is negative or positive. On the other hand, χ_1 and χ_2 have stabilizing effects on one mode while they have destabilizing effects on the other mode, irrespective of their signs. A negative μ_2 has a stabilizing effect on a_1, while it has a stabilizing or destabilizing effect on a_2, depending on whether $(\omega_1 + \omega_2)a_1{}^2 + \omega_2 a_2{}^2$ is negative or positive. On the other hand, a positive μ_2 has a destabilizing effect on a_1, while it has a stabilizing or destabilizing effect on a_2 depending on whether $(\omega_1 + \omega_2)a_1{}^2 + \omega_2 a_2{}^2$ is positive or negative. For a positive p, the asymmetry has a stabilizing or destabilizing effect on a_1/a_2 depending on whether $(\omega_1 + p)\mu_2 + 2\chi_2$ is negative or positive/$(\omega_2 + p)\mu_2 + 2\chi_2$ is positive or negative.

The motion in this case is given by Eqs. (13) and (17) along with the solution of Eq. (18) which is given by

$$\xi_3 = \frac{Q_3}{(\omega_1 - p)(p - \omega_2)}\exp(ipT_0) + \sum_{n=1}^{2}\left\{\frac{1}{2}\Delta^2 m_n A_n{}^2\bar{A}_s\exp[i(2\omega_n - \omega_s)T_0] + \frac{m_n A_n A_s\bar{A}_3}{(\omega_1 - p)(p - \omega_2)}\exp[i(\omega_2 + \omega_1 - p)T_0] + \frac{m_n A_n{}^2\bar{A}_3}{(\omega_n - p)(p - 2\omega_n + \omega_s)}\exp[i(2\omega_n - p)T_0] + \frac{(m_n + m_3)A_n\bar{A}_s A_3}{(\omega_s - p)(p + \omega_n - 2\omega_s)}\exp[i(p - \omega_s + \omega_n)T_0] + \frac{m_3\bar{A}_n A_3{}^2}{2(p - \omega_n)(\omega_1 + \omega_2 - 2p)}\exp[i(2p - \omega_n)T_0]\right\} \tag{21}$$

where $s = 2$ when $n = 1$ and $s = 1$ when $n = 2$.

The transient solutions of the system of perturbation equations are in good agreement with the numerical solution of Eq. (8a) and six-degree-of-freedom calculations. Moreover, the trends predicted previously are substantiated by the numerical solutions.[9]

For positive p, ω_1 is positive and ω_2 is negative. The denominators of all but the second term vanish at $p = \omega_1$ while the denominator of the fourth vanishes at $p = 2\omega_1 - \omega_2 = 3\omega_1/(1 + I_x/I)$. The former case will occur at roll resonance and will be discussed later. The latter case, a subharmonic resonance due entirely to the nonlinearities, was first discussed by Murphy[1] and will be analyzed next. The denominator of the last term will vanish when $p \to 0$ which also requires special treatment and is discussed later.

The $p = 2\omega_1 - \omega_2$ and $p = 0$ Cases

To examine the case where $p \approx 2\omega_1 - \omega_2$, we let

$$p = 2\omega_1 - \omega_2 + \varepsilon^2\tilde{\sigma}, \quad \tilde{\sigma} = 0(1) \tag{22}$$

where $\tilde{\sigma}$ is the detuning parameter. Equation (22) is substituted into the right-hand side of Eq. (18). To eliminate secular terms in the solution of ξ_3, Eqs. (19) are modified to[9]

$$da_n/dT_2 = (\lambda_{1n} + \lambda_{2n}a_n{}^2 + \lambda_{3n}a_s{}^2 + \lambda_{4n}a_3{}^2)a_n - (2)^{2-n}\Delta[\gamma\sin\tilde{\psi} + (-1)^n(\chi_2 + \omega_1\mu_2)\cos\tilde{\psi}]a_1 a_s a_3 \tag{23a}$$

$$d\theta_n/dT_2 = (-1)^n\gamma\Delta(a_n{}^2 + 2a_s{}^2 + 2a_3{}^2) + (-2)^{2-n}\Delta[\gamma\cos\tilde{\psi} + (-1)^{n+1}(\chi_2 + \omega_1\mu_2)\sin\tilde{\psi}]a_s{}^2 a_3/a_2 \tag{23b}$$

§ It is recognized that other definitions are possible if one wishes to include damping terms.

where

$$\tilde{\psi} = 2\theta_1 - \theta_2 - \tilde{\sigma}T_2 - \phi_0 \quad (24)$$

The solution for ξ_3 is given by Eq. (21) except that the terms singular in $(p-2\omega_1+\omega_2)^{-1}$ are missing. The solutions of these perturbation equations are in good agreement with the numerical solution of Eq. (8a) for the $p = 2\omega_1 - \omega_2$ case.[9]

To examine the case of small roll rates, we let

$$(2 - I_x/I)p = \varepsilon^2\hat{\sigma}, \quad \hat{\sigma} = 0(1) \quad (25)$$

where $\hat{\sigma}$ is the detuning parameter. Equation (25) is substituted into the right-hand side of Eq. (18). In this case, χ_1 and $\chi_2 \approx 0$, $\omega_2 \approx -\omega_1$, and Eqs. (19) are modified to[9]

$$da_n/dT_2 = (\mu_1 + \mu_2 a_n{}^2 + \mu_2 a_3{}^2)a_n/2 + (-1)^n(m_3 a_s a_3{}^2/2\omega_0)\sin\hat{\psi} \quad (26)$$

$$d\hat{\psi}/dT_2 = -\hat{\sigma} + (\gamma + m_3 a_3{}^2 a_1{}^{-1} a_2{}^{-1}\cos\hat{\psi})(a_1{}^2 - a_2{}^2)/2\omega_0 \quad (27)$$

where

$$\hat{\psi} = \theta_1 + \theta_2 - \hat{\sigma}T_2 - 2\phi_0 \quad (28)$$

With the secular producing terms eliminated, ξ_3 is given by (21) except that the singular terms are missing. Thus, the system of Eqs. (13), (17), and the modified (21) are valid approximations to the full nonlinear system when $p \approx 0$. Conclusions on the stabilizing or destabilizing effects of certain parameters can be drawn from Eqs. (26) and (27).

The $p = \omega_1$ Case

In this case we assume a weak excitation where the trim moment coefficient is written as $K = \varepsilon^2 k$ where $k = 0(\omega_0{}^2)$. To examine the behavior near roll resonance when $p \approx \omega_1$, a detuning parameter σ, is introduced in the equation for p by

$$p = \omega_1 + \varepsilon^2\sigma, \quad \sigma = 0(1) \quad (29)$$

The asymmetry excitation is expressed as

$$\varepsilon K \exp i(pT_0 + \phi) = \varepsilon^3 \Lambda \exp i\omega_1 T_0, \quad \Lambda = k \exp i(\sigma T_2 + \phi_0) \quad (30)$$

The representation of Eqs. (30) makes the right-hand-side of Eq. (15) vanish while the right-hand-side of Eq. (16) is modified to contain the term $\Lambda \exp(i\omega_1 T_0)$. Consequently, to eliminate secular producing terms in the modified Eq. (16), we get

$$da_n/dT_2 = (\lambda_{1n} + \lambda_{2n} a_n{}^2 + \lambda_{3n} a_s{}^2)a_n + (2-n)k\Delta \sin\psi \quad (31a)$$

$$d\theta_n/dT = (-1)^n\gamma\Delta(a_n{}^2 + 2a_s{}^2) - (2-n)k\Delta a_1{}^{-1}\cos\psi \quad (31b)$$

where $s = 2, 1$ when $n = 1, 2$ and

$$\psi = \sigma T_2 - \theta_1 + \phi_0 \quad (32)$$

For the weak excitation case considered, ξ_1 and ξ_3 are given by Eqs. (17) and (21) with $a_3 = 0$. The effect of k is implicit through the solutions for a_1 and θ_1. Comparisons between the perturbation solution and the numerical integration of Eq. (8a) are undetectable when superposed. It appears, therefore, that the perturbation solution is a valid approximation in the neighborhood of roll resonance provided the roll rate and excitation are constant. The conclusions drawn from the previous discussions above concerning the stabilizing or destabilizing effects of χ_1, χ_2 and other parameters are substantiated[9] near roll resonance in this case by numerical integration of Eq. (8a).

The steady-state solutions near roll resonance correspond to the stationary solutions of Eqs. (31); that is, they correspond to the solutions of

$$a_{20}(\lambda_{12} + \lambda_{22} a_{20}{}^2 + \lambda_{32} a_{10}{}^2) = 0 \quad (33)$$

$$\lambda_{11} + \lambda_{21} a_{10}{}^2 + \lambda_{31} a_{20}{}^2 = -k\Delta a_{10}{}^{-1}\sin\psi_0 \quad (34)$$

$$\sigma + \gamma\Delta(a_{10}{}^2 + 2a_{20}{}^2) = -k\Delta a_{10}{}^{-1}\cos\psi_0 \quad (35)$$

Since $d\psi/dT_2 = 0$ in this case, Eq. (32) gives $d\theta_1/dT_2 = \sigma$ and

$$\xi = \varepsilon[A_1 \exp(i\omega_1 T_0) + A_2 \exp(i\omega_2 T_0)] + 0(\varepsilon^3) \quad (36)$$

or

$$\xi = \varepsilon\{a_{10}\exp(ipt) + a_{20}{}^2 \exp i[\omega_2 + \varepsilon^2\gamma\Delta(2a_{10}{}^2 + a_{20}{}^2)]t\} + 0(\varepsilon^3) \quad (37)$$

Thus the nutation component of the angle of attack response is synchronized to the excitation frequency p even for a slight asymmetry. The nonlinearity adjusts the nutation frequency so that it becomes exactly equal to the roll rate (perfect roll resonance). Consequently, Murphy's[1] separation of the rolling and nutation response is questionable.

We investigate the solutions of Eqs. (33–35) and their stability using Eqs. (31) when the Magnus and damping terms are negligible as is the case at high altitudes. The general case of $\mu_{1,2}$ and $\chi_{1,2}$ not equal to zero is treated in Ref. 9. Under the conditions of $\mu_{1,2} \approx 0$ and $\chi_{1,2} \approx 0$, Eq. (31) gives $a_2 = a_{20} =$ const., while

$$da_1/dT_2 = k\Delta \sin\psi \quad (38a)$$

$$d\psi/dT_2 = \sigma + \gamma\Delta(a_1{}^2 + 2a_{20}{}^2) + (k\Delta/a_1)\cos\psi \quad (38b)$$

The steady-state solution in this case is given by

$$\psi_0 = m\pi \quad \text{with integer} \quad m \quad (39a)$$

$$a_{10}[\sigma + \gamma\Delta(a_{10}{}^2 + 2a_{20}{}^2)] = k\Delta\cos\psi \quad (39b)$$

The latter equation is a cubic in a_{10} which can be solved to give a_{10} as a function of σ, γ, a_{20}, ω_1, and ω_2. The stability of this solution can be investigated by substituting for a_1 and ψ from $a_1 = a_{10} + (\delta a_1)\exp(sT_2)$ and $\psi = \psi_0 + (\delta\psi)\exp(sT_2)$ into Eqs. (38), using Eqs. (39), and neglecting nonlinear perturbation quantities. The result is

$$s^2 = k\Delta a_{10}{}^{-1}[\sigma + \gamma\Delta(3a_{10}{}^2 + 2a_{20}{}^2)]\cos\psi_0 \quad (40)$$

Hence the steady-state solution is unstable if

$$[\sigma + \gamma\Delta(3a_{10}{}^2 + 2a_{20}{}^2)]\cos\psi_0 > 0 \quad (41)$$

otherwise, it is stable.

4. The Variable Roll Rate, Nonresonant Case

Although the constant roll rate case is useful for interpreting wind-tunnel data, it does not represent the actual flight situation. Therefore, the complete set of Eqs. (8) must be solved. To determine an approximate solution to Eqs. (8) using the generalized version of the method of multiple scales,[4] we make use of the fact that actual flight test data and six-degree-of-freedom numerical calculations show that there are at least four time scales; a slow time scale $T_2 = \varepsilon^2 t$ characterizing the variation of the dynamic pressure and stability derivatives, and three fast scales characterizing the nutation, precession, and forced components of the angle of attack. Thus, we assume expansions of the form

$$\xi(t;\varepsilon) = \varepsilon\xi_1(\eta_1, \eta_2, \phi, T_2) + \varepsilon^3\xi_3(\eta_1, \eta_2, \phi, T_2) + \ldots \quad (42)$$

$$p(t;\varepsilon) = p_0(T_2) + \varepsilon^2 p_2(\eta_1, \eta_2, \phi, T_2) + \ldots \quad (43)$$

where

$$d\eta_n/dt = \omega_n, \quad \omega_{1,2} = (p_0 I_x/2I) \pm [(p_0 I_x/2I)^2 + \omega_0{}^2]^{1/2} \quad (44)$$

with $\omega_1 = \omega_1(T_2)$ and $\omega_2 = \omega_2(T_2)$ the nutation and precession frequencies. In terms of these variables, the time derivatives are transformed according to

$$d/dt = \mathscr{H} + \varepsilon^2\, \partial/\partial T_2 \quad (45a)$$

$$d^2/dt^2 = \mathscr{H}^2 + 2\varepsilon^2\, \partial\mathscr{H}/\partial T_2 + \varepsilon^4\, \partial^2/\partial T_2{}^2 \quad (45b)$$

where

$$\mathscr{H} = \omega_1\, \partial/\partial\eta_1 + \omega_2\, \partial/\partial\eta_2 + p\, \partial/\partial\phi \quad (46)$$

The conditions that Eqs. (42) and (43) are uniformly valid for all t are equivalent to the conditions

$$\xi_3/\xi_1 < \infty \quad \text{and} \quad p_2/p_0 < \infty \quad \text{for all} \quad t$$

Thus, uniformly valid expansions are obtained by determining the functional dependence of ξ_1 and p_0 on T_2 in order to satisfy the preceding conditions.

Substituting Eqs. (42–46) into Eqs. (8) and equating coefficients of like powers of ε we get equations for the determination of ξ_i and p_i. The first-order problem (i.e., coefficient of ε) is

$$\mathscr{L}(\xi_1) = K\exp[i(\phi + \phi_0)] \quad (47)$$

where

$$\mathscr{L} = \left(\omega_1 \frac{\partial}{\partial \eta_1} + \omega_2 \frac{\partial}{\partial \eta_2} + p_0 \frac{\partial}{\partial \phi}\right)^2 - i\frac{p_0 I_x}{I}\left(\omega_1 \frac{\partial}{\partial \eta_1} + \omega_2 \frac{\partial}{\partial \eta_2} + p_0 \frac{\partial}{\partial \phi}\right) + \omega_0^2 \quad (48)$$

The solution of this problem is taken to be

$$\xi_1 = A_1(T_2)\exp(i\eta_1) + A_2(T_2)\exp(i\eta_2) + A_3(T_2)\exp(i\phi) \quad (49)$$

where as in Sec. 3, $A_3 = K(\omega_1 - p_0)^{-1}(p_0 - \omega_2)^{-1}\exp(i\theta_0)$ and $A_n = a_n \exp(i\theta_n)$ with real amplitudes a_n and phases θ_n ($n = 1, 2$).

The equation for p_2 becomes

$$\mathscr{H}_0(p_2) = -dp_0/dT_2 + v_0 + v_2 p_0 + v_1 \operatorname{Imag}(A_3) + \sum_{n=1}^{2} v_1 a_n \sin(\eta_n - \phi + \theta_n) \quad (50)$$

where $\mathscr{H}_0$ is the same as $\mathscr{H}$ except p is replaced by p_0. The particular solution of this equation is unbounded as η_1, η_2, or $\phi \to \infty$ (i.e., $t \to \infty$) unless

$$dp_0/dT_2 = v_0 + v_2 p_0 + (\omega_1 - p_0)^{-1}(p_0 - \omega_2)^{-1} v_1 K \sin\phi_0 \quad (51)$$

Then the solution of Eq. (50) becomes

$$p_2 = v_1 \sum_{n=1}^{2} a_n (p_0 - \omega_n)^{-1} \cos(\eta_n - \phi + \theta_n) \quad (52)$$

With ξ_1 and p_2 known, the equation for ξ_3 becomes

$$\mathscr{L}(\xi_3) = \tilde{Q}_3 \exp(i\phi) + \sum_{n=1}^{2} [\tilde{Q}_n \exp(i\eta_n) + m_n A_n^2 \bar{A}_s \times \exp i(2\eta_n - \eta_s) + m_n A_n^2 \bar{A}_3 \exp i(2\eta_n - \phi) + m_n A_1 A_2 \bar{A}_3 \times \exp i(\eta_1 + \eta_2 - \phi) + (m_n + m_3) A_n \bar{A}_s A_3 \exp i(\phi + \eta_n - \eta_s) + (m_3 \bar{A}_n A_3^2 + \tilde{Q}_{4n}) \exp i(2\phi - \eta_n)] \quad (53)$$

where the $\tilde{Q}$'s are defined in Appendix A and $s = 2, 1$ when $n = 1, 2$. The particular solution of Eq. (53) contains secular terms of the form $\eta_s \exp(i\eta_s)$ which make ξ_3/ξ_1 unbounded as $t \to \infty$. To get a uniformly valid expansion, A_1 and A_2 are chosen so that the secular terms vanish; that is $\tilde{Q}_1 = \tilde{Q}_2 = 0$. The solution for ξ_3 from Eq. (53) is the same as Eq. (21) except that pT_0, $\omega_1 T_0$, $\omega_2 T_0$ are interpreted as ϕ, η_1, η_2 respectively, Q_3 is replaced by $\tilde{Q}_3$ and the following term is added under the summation sign

$$\tfrac{1}{2}\tilde{Q}_{4n}(\omega_1 + \omega_2 - 2p_0)^{-1}(p_0 - \omega_n)^{-1} \exp i(2\phi - \eta_n) \quad (54)$$

The details can be found in Ref. 10.

The condition $\tilde{Q}_1 = \tilde{Q}_2 = 0$ gives

$$\frac{da_n}{dT_2} = (\tilde{\lambda}_{1n} + \lambda_{2n} a_n^2 + \lambda_{3n} a_s^2 + \lambda_{4n} a_3^2) a_n + (-1)^n a_n \Delta\left[\frac{1}{2} - \left(1 - \frac{I_x}{I}\right)\frac{p_0}{p_0 - \omega_n}\right]\sin\phi_0 \quad (55a)$$

$$\frac{d\theta_n}{dT_2} = (-1)^n \Delta\gamma(a_n^2 + 2a_s^2 + 2a_3^2) + (-1)^{n+1} v_1 a_3 \Delta\left[\frac{1}{2} - \left(1 - \frac{I_x}{I}\right)\frac{p_0}{p_0 - \omega_n}\right]\cos\phi_0 \quad (55b)$$

where $s = 2, 1$ when $n = 1, 2$, and

$$\tilde{\lambda}_{1n} = \lambda_{1n} + (-1)^n \Delta\omega_n', \quad \omega_n' = d\omega_n/dT_2 \quad (56)$$

with the other λ's defined in Eqs. (20). These equations are similar to Eqs. (19) derived in Sec. 3 except for the presence of ω_n' and v_1 terms. As discussed in Sec. 3, the solution breaks down as p_0 approaches ω_1, 0, and $2\omega_1 - \omega_2$. The first case is the roll resonance case and is analyzed in Sec. 5. The last case corresponds to $p_0 \approx 3\omega_1$ for slender bodies and leads to subharmonic resonances. Solutions valid near $p_0 = 3\omega_1$ and $p = 0$ are presented later in this section.

The discussion regarding the effects of $\mu_{1,2}$, $\chi_{1,2}$ and K is contained in Sec. 3. The effects of the roll acceleration $p_0' = dp_0/dT_2$ and the rate of the critical frequency $\omega_0' = d\omega_0/dT_2$ on the stability of the body motion are given implicitly by ω_1' and ω_2'. Equation (44) shows that both ω_1' and ω_2' are positive/negative depending on whether p_0' is positive/negative. Hence, Eqs. (55) and (56) show a positive/negative roll acceleration has a stabilizing/destabilizing effect on a_1 and a destabilizing/stabilizing effect on a_2. On the other hand, $\omega_1' > 0$ while $\omega_2' < 0$ if $\omega_0' > 0$ (above maximum dynamic pressure). Therefore, a positive/negative ω_0' has a stabilizing/destabilizing effect on both modes. This means a body may pass through roll resonance before maximum dynamic pressure successfully without large amplification if conditions are *proper* but it may not successfully pass through roll resonance after maximum dynamic pressure if a low-altitude encounter (i.e., a disturbance) is experienced.

The solution of the system of perturbation equations, are in good agreement with the numerical solution of Eqs. (8) as shown in Ref. 10. However, the perturbation solution breaks down rapidly as roll resonance is approached. If $0.7\omega_0 < p < 1.3\omega_0$ the agreement is not very good, and the solution, to be derived in Sec. 5, valid near roll resonance must be used. If $0.99(2\omega_1 - \omega_2) < p < 1.01(2\omega_1 - \omega_2)$ or $p < 0.01\omega_0$ then the solutions, derived later in this section, valid near these singularities, must be used.

The $p = 2\omega_1 - \omega_2$ and $p = 0$ Cases

To examine the case where $p \approx 2\omega_1 - \omega_2$ we follow the treatment of Sec. 3 and obtain the following modifications to Eqs. (55)

$$da_n/dT_2 = \Xi_{1n} - (2)^{2-n}\Delta[\gamma \sin\tilde{\Gamma} + (-1)^n(\chi_2 + \omega_1\mu_2)\cos\tilde{\Gamma}] a_1 a_s a_3 \quad (57a)$$

$$d\theta_n/dT_2 = \Xi_{2n} + (-2)^{2-n}\Delta[\gamma\cos\tilde{\Gamma} + (-1)^{n+1}(\chi_2 + \omega_1\mu_2)\sin\tilde{\Gamma}] a_s^2 a_3/a_2 \quad (57b)$$

where

$$\tilde{\Gamma} = 2\eta_1 - \eta_2 - \phi + 2\theta_1 - \theta_2 - \phi_0 \quad (58)$$

and Ξ_{1n} and Ξ_{2n} stand for the right-hand sides of Eqs. (55a) and (55b), respectively. The solution for ξ_3 is given by the modified Eq. (21) with Eq. (54), except that the terms singular in $(p - 2\omega_1 + \omega_2)^{-1}$ are missing. The expressions given in Ref. 10 for this case are incorrect.

To examine the case of small roll rates, χ_1 and $\chi_2 \approx 0$, $\omega_1 \approx \omega_0$ and $\omega_2 \approx -\omega_0$. Equations (55) are modified to

$$2\frac{da_n}{dT_2} = [\mu_1 + \mu_2 a_n^2 + \mu_2 a_3^2 + (-1)^n \omega_n'/\omega_0] a_n - \gamma\frac{a_s a_3^2}{\omega_0}\sin\tilde{\Gamma} + (-1)^n \frac{v_1 a_n a_3}{2\omega_0}\sin(\tilde{\Gamma} + \phi_0) \quad (59)$$

$$\frac{d\theta_n}{dT_2} = \frac{(-1)^n\gamma}{2\omega_0}(a_n^2 + 2a_s^2 + 2a_3^2) + (-1)^{n+1} v_1 \frac{a_3}{4\omega_0} + (-1)^n\gamma\frac{a_s a_3^2}{2a_n\omega_0}\cos\tilde{\Gamma} + (-1)^n\frac{v_1 a_s a_3}{4a_n\omega_0}\cos(\tilde{\Gamma} + \phi_0) \quad (60)$$

where

$$\tilde{\Gamma} = \eta_2 + \eta_1 - 2\phi + \theta_2 + \theta_1 - 2\phi_0 \quad (61)$$

For constant roll rate and constant dynamic pressure, Eqs. (59) and (60) reduce to those obtained in Sec. 3. The combination of a c.g. offset and a trim angle of attack, characterized by the term proportional to v_1, tends to increase the amplitude of one mode and decrease the other mode as evident from Eq. (59). In the absence of a trim angle of attack (i.e., $a_3 = 0$), the two modes of oscillation are decoupled; otherwise, they are coupled.

5. Roll Resonance

In this case $p_0 \approx \omega_1$, and we assume $K = \varepsilon^2 k$ and seek expansions of the form

$$\xi(t;\varepsilon) = \varepsilon\xi_1(\eta_1, \eta_2, T_2) + \varepsilon^3\xi_3(\eta_1, \eta_2, T_2) + \ldots \quad (62)$$

$$p(t;\varepsilon) = p_0(T_2) + \varepsilon^2 p_2(\eta_1, \eta_2, \phi, T_2) + \ldots \quad (63)$$

where $d\eta_n/dt = \omega_n$ and ω_n are given by Eq. (44). Substituting these expansions into Eqs. (8) and equating coefficients of like powers of ε, we have for the first-order problem of Eq. (8a),

$$\mathscr{L}(\xi_1) = 0 \tag{64}$$

where $\mathscr{L}$ is defined by Eq. (48). The solution to this problem is

$$\xi_1 = A_1 \exp(i\eta_1) + A_2 \exp(i\eta_2) \tag{65}$$

Then the first-order problem of Eq. (8c) is

$$\mathscr{H}_0(p_2) = -dp_0/dT_2 + v_0 + v_2 p_0 + v_1[a_1 \sin(\eta_1 - \phi + \theta_1) + a_2 \sin(\eta_2 - \phi + \theta_2)] \tag{66}$$

where the effect of the asymmetry is implicit in the solution of a_1. This implies, of course, that k must be a slowly varying function of time.

Since $p_0 \approx \omega_1$, $(\eta_1 - \phi)$ is a slowly varying function of time, and we consider it to be a function of T_2. Now, the solution of Eq. (66) contains terms which tend to infinity as η_1, η_2, or $\phi \to \infty$ (i.e., $t \to \infty$), thereby invalidating our expansion for long times, unless

$$dp_0/dT_2 = v_0 + v_2 p_0 + v_1 a_1 \sin(\eta_1 - \phi + \theta_1) \tag{67}$$

Then p_2 becomes

$$p_2 = a_2(p_0 - \omega_2)^{-1} \cos(\eta_2 + \theta_2 - \phi) \tag{68}$$

With ξ_1 and p_2 known, the third-order problem of Eq. (8a) becomes

$$\mathscr{L}(\xi_3) = \sum_{n=1}^{2} [\hat{Q}_n \exp(i\eta_n) + m_n A_n^2 \bar{A}_s \exp i(2\eta_n - \eta_s)] \tag{69}$$

where

$$\hat{Q}_n = \tilde{Q}_n(a_3 = 0) + (2-n)k \exp i(\phi + \phi_0 - \eta_1) \tag{70}$$

Secular terms will be eliminated if $\hat{Q}_n = 0$ which leads to

$$da_n/dT_2 = (\tilde{\lambda}_{1n} + \lambda_{2n} a_n^2 + \lambda_{3n} a_s^2)a_n + (2-n)k\Delta \sin\Gamma \tag{71a}$$

$$d\theta_n/dT_2 = (-1)^n \gamma\Delta(a_n^2 + 2a_s^2) - (2-n)k\Delta a_1^{-1} \cos\Gamma \tag{71b}$$

where the λ's are defined in Eqs. (20) and (56) and

$$\Gamma = \phi - \eta_1 - \theta_1 + \phi_0 \tag{72}$$

Introducing the detuning parameter σ defined by

$$p_0 = \omega_1 + \varepsilon^2 \sigma \tag{73}$$

we have

$$d\Gamma/dT_2 = \sigma + \gamma\Delta(a_1^2 + 2a_2^2) + k\Delta a_1^{-1} \cos\Gamma \tag{74}$$

and the solution of Eq. (69) is

$$\xi_3 = -\frac{1}{2} \sum_{n=1}^{2} m_n \Delta^2 A_n^2 \bar{A}_s \exp i(2\eta_n - \eta_s) \tag{75}$$

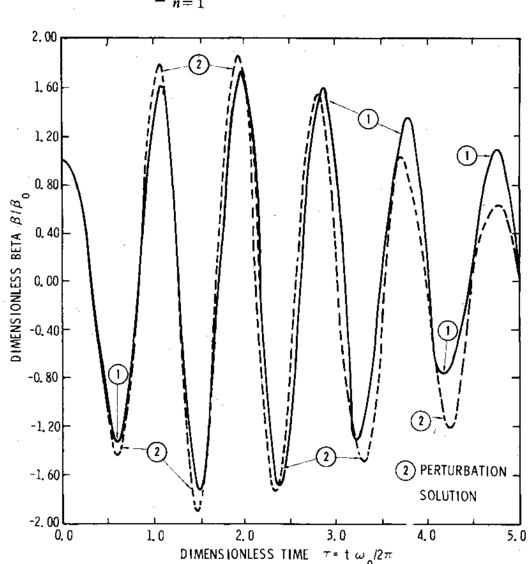

Fig. 2 Variation of the yaw angle of attack with time [$\omega_0 = 30.7$, $\beta_0 = 10°$, $\varepsilon = 0.174$, $k = 33423$, $v_0 = 50$, $v_1 = 60$, $v_2 = -1.4$, $\mu_1 = 14.6$, $\mu_2 = -30.7$, $\chi_1 = -942\, p(t)/p(t=0)$, $\chi_2 = 0$, $\gamma = -942$, **and initial** $\sigma = 100$].

Figure 2 compares our analytical results with the numerical solution of Eqs. (8) for the case of nonlinear aerodynamics. The value of $K = 1007.8$ used in these and subsequent figures is the large trim excitation that is used in Sec. 4. That is, $\varepsilon K/\omega_0^2 = 0(1)$ instead of $\varepsilon K/\omega_0^2 = 0(\varepsilon^2)$ as postulated in the roll resonance analysis. Furthermore, a value of $\sigma = 100$ was also used. These extreme values of K and σ were used only to exaggerate the differences between the two solution techniques and establish the fact that the roll resonance solution is valid beyond the intended range. The physical characteristics of the vehicle used to generate the different coefficients are given in Appendix B. For linear aerodynamics, the two solutions compare precisely. When the nonlinear moments are added, differences occur but the trends are the same. Figure 3 is a comparison of roll calculations using the two techniques. For linear aerodynamics there are no detectable differences. For the nonlinear aerodynamics case, some small differences occur at $\tau = 5$ which are of the order of 1%. The effect of the nonlinear damping and restoring moments is to reduce the amplitude and rate of growth of the angle-of-attack oscillations which in turn reduces the term $v_1 a_1 \sin(\eta_1 + \theta - \phi)$ in Eq. (67) which reduces dp_0/dT_2.

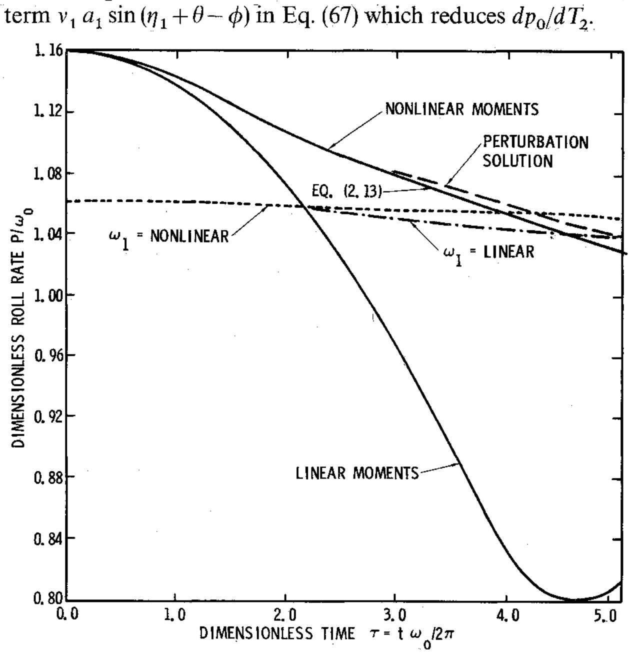

Fig. 3 Variation of the roll rate with time (same conditions as Fig. 2).

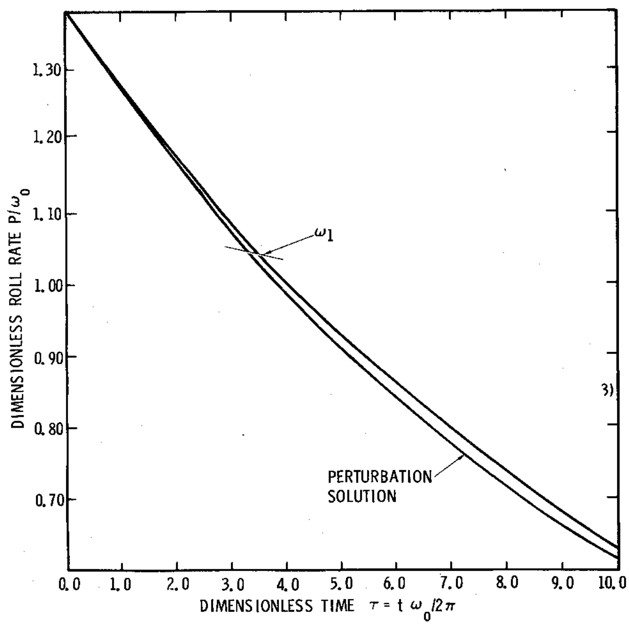

Fig. 4 Variation of the roll rate with time [$\varepsilon = 0.174$, $k = 1007.8$, $\chi_1 = -545\ p(t)/p(t=0)$, $\chi_2 = 0$, $\gamma = -987$, $\mu_2 = -31.4$, **and initial** $\sigma = 300$].

Figure 4 compares the two techniques for the case of $\varepsilon k/\omega_0^2 = 0(\varepsilon^2)$ and initial $\sigma \approx 300$. Thus, the trim moments are ordered properly but the range of the roll rate is extended. The error in roll rate at $\tau = 10$ when $\sigma = -434$ is 2.6%. The initial conditions are comparable to a match of solutions using the analysis of Sec. 4 away from roll resonance.

We can conclude then that 1) if $\varepsilon k/\omega_0^2 = 0(\varepsilon^2)$ or less, the perturbation solution, valid near roll resonance, is in very good agreement with the numerical solution of the full problem. This is true for the nonlinear aerodynamics case and the ranges of roll rates required to match with the solution away from roll resonance; 2) if $\varepsilon k/\omega_0^2 = 0(1)$, the perturbation solution is in very good agreement with the numerical solution of the full problem for linear aerodynamics and is sufficiently accurate to predict the trends in the solutions for nonlinear aerodynamics.

6. Conclusions

The results of the preceding two sections show that the roll rate is given by

away from roll resonance

$$\frac{dp_0}{dT_2} = v_0 + v_2 p_0 + v_1 K \sin\phi_0/(\omega_1 - p_0)(p_0 - \omega_2) \quad (76)$$

near roll resonance

$$v_0 + v_2 p_0 + v_1 a_1 \sin(\eta_1 - \phi + \theta_1)$$

In Eq. (76), v_0 characterizes the pure rolling moment and v_1 characterizes the c.g. offset symmetry. The effect of trim moments is contained in a_1 near roll resonance. If ω_1 is replaced by p_0 in the coefficients multiplying $\sin\Gamma$ and $\cos\Gamma$ in Eqs. (71), the results in Sec. 5 are valid near and away from roll resonance. Near resonance $p_0 \approx \omega_1$ and the modified equations are the same as Eqs. (71) within an error of $0(\varepsilon^2)$.

Away from resonance, ϕ is away from η_1, and Eqs. (71) show that a_1 and θ_1 are composed of a short period term (fast varying) and a long period term (slowly varying). To separate these parts, we let $a_1 = a_1^* + \hat{a}_1$, $\theta_1 = \theta_1^* + \hat{\theta}_1$ where a_1^* and θ_1^* are slowly varying while the other components contain the fast varying parts. Substituting these forms into the modified Eqs. (71) and separating the fast and slowly varying parts, we get

$$d\hat{a}_1/dt = \varepsilon^2 k(p_0 - \omega_2)^{-1} \sin(\phi - \eta_1 - \theta_1 + \phi_0) \quad (77a)$$

$$d\hat{\theta}_1/dt = -\varepsilon^2 k(p_0 - \omega_2)^{-1} a_1^{*-1} \cos(\phi - \eta_1 - \theta_1 + \phi_0) \quad (77b)$$

$$da_1^*/dT_2 = (\lambda_{11} + \lambda_{21} a_1^{*2} + \lambda_{22} a_2^{*2}) a_1^* \quad (78a)$$

$$d\theta_1^*/dT_2 = -\gamma\Delta(a_1^{*2} + 2a_2^{*2}) \quad (78b)$$

Approximate solutions to Eqs. (77) are

$$\hat{a}_1 = -\varepsilon^2 k(p_0 - \omega_1)^{-1}(p_0 - \omega_2)^{-1} \cos\Gamma \quad (79a)$$

$$\hat{\theta}_1 = -\gamma\varepsilon^2 k(p_0 - \omega_1)^{-1}(p_0 - \omega_2)^{-1} \sin\Gamma \quad (79b)$$

Letting $a_1 = a_1^* + \hat{a}_1$ and $\theta_1 = \theta_1^* + \hat{\theta}_1$ in Eqs. (65) and (66) and expanding for small ε, we get the results of Sec. 4 for the case of small excitation.

The roll resonance condition is defined by $p_0 = \omega_1$ which leads from Eq. (44) to the two critical roll rates

$$p_{cr} = \pm\omega_0/(1 - I_x/I)^{1/2} \quad (80)$$

For slender bodies, the critical roll rates are $p_{cr} \approx \pm\omega_0$.

Equation (76) shows that the parameter v_2 leads to the damping of the roll rate, while the parameter v_0 leads to roll up or roll down depending on whether it is positive or negative. Thus, v_0 may produce roll resonance, but it cannot by itself produce a stable roll resonance.

The parameter responsible for conditions of stable roll resonance is $v_1 \sin\phi_0$ which is a combination of a c.g. offset and an out-of-plane trim. The first of Eqs. (76) shows that if $v_1 \sin\phi_0$ is positive, p_0 tends toward ω_1 if $p_0 > \omega_2$ and tends away from ω_1 if $p_0 < \omega_2$. Thus, a necessary condition for a roll lock-in on the critical roll rate ω_0 is $v_1 \sin\phi_0 > 0$ and an initial roll rate greater than ω_2. In addition to this condition, the time variation of the roll rate must be larger in magnitude than the time variation of p_{cr} for lock-in (stable roll resonance); that is

$$|v_0 + v_2 p_0 + v_1 a_1 \sin(\phi_0 + \theta_1)| \geqq |d\omega_0/dT_2| \quad (81)$$

Since ω_0 is a function of the dynamic pressure, Eq. (81) yields a condition for stable roll resonance in terms of the rolling moment parameters, the ballistic coefficient, and the flightpath angle.

Appendix A

The expressions for $Q_{1,2,3}$ and $m_{1,2,3}$ for Eq. (18) are given as

$$Q_n = i(p_0 I_x I^{-1} - 2\omega_n) dA_n/dT_2 + i[\omega_n \mu_1 + \chi_1 - im_n a_n^2 - i(m_1 + m_2)a_s^2 - i(m_n + m_3)a_3^2]A_n \quad (A1)$$

$$Q_3 = [i(p_0 \mu_1 + \chi_1) + (m_1 + m_3)a_1^2 + (m_2 + m_3)a_2^2 + m_3 a_3^2]A_3 \quad (A2)$$

$$m_n = i\omega_n \mu_2 + i\chi_2 + \gamma \quad (A3)$$

$$m_3 = ip_0 \mu_2 + i\chi_2 + \gamma \quad (A4)$$

where $s = 2, 1$ when $n = 1, 2$. The expressions for $\tilde{Q}_{1,2,3}$ and $\tilde{Q}_{4n}$ for Eq. (53) are given as

$$\tilde{Q}_n = Q_n - i\omega_n' A_n + v_1(p_0 - \omega_n)^{-1}[\omega_n/2 + (1/2 - I_x/I)p_0]A_n A_3 \quad (A5)$$

$$\tilde{Q}_3 = Q_3 - ip_0' A_3 \quad (A6)$$

$$\tilde{Q}_{4n} = v_1(p_0 - \omega_n)^{-1}[-\omega_n/2 + (3/2 - I_x/I)p_0]\bar{A}_n A_3 \quad (A7)$$

Appendix B

The physical characteristics of the vehicle chosen for the analysis are tabulated in Table 1 using standard notation.

Table 1 Physical characteristics of the vehicle

Mass:	$m = 5.122$ slugs
Moments of inertia:	$I = 4.31$ slugs ft^2
	$I_x = 0.43$ slugs ft^2
Area:	$A = 1.485$ ft^2
Diameter:	$d = 1.375$ ft
Normal force coefficient:	$C_{N_\alpha} = 1.934$/rad
Pitching moment coefficient:	$C_{M_\alpha} = -0.3314$/rad
Ballistic coefficient:	Beta = 1500 lb/ft^2
Damping moment coefficient:	$C_{m_q} = 4$/rad
Pure rolling moment coefficient:	$C_{l_0} = 10^5$
Roll damping coefficient:	$C_{l_p} = -0.2$/rad
Trim angles:	$\alpha_T = 1/3°$
	$\beta_T = 0$
c.g. offset:	$\delta = 0.03$ in.

At an altitude of 100,000 ft with a velocity of 19,700 fps, the dynamic pressure is 6,290 psf and the parameters introduced in the differential equation are calculated as follows:

$$\omega_0 = 10\pi \text{ rad/sec}, \quad \varepsilon^2\mu_1 = QAd(dC_{m_q}/2V) = 0.416/\text{sec}$$

$$\varepsilon^2 v_0 = (QAd/I_x)C_{l_0} = 0.3/\text{sec}^2,$$

$$\varepsilon^2 v_2 = (QAd/I_x)C_{l_p}(d/2v) = -0.41/\text{sec}$$

$$\varepsilon v_1 = (QAd/I_x)C_{N_\alpha}(\delta/d) = 10.5/\text{sec}^2, \quad \varepsilon K = 175/\text{sec}^2$$

The unusual positive sign on C_{m_q} is chosen to produce growing angle of attack oscillations in order to illustrate any differences that may occur between the perturbation solution and the numerical solution of the full problem. In the case of strongly damped solutions where C_{m_q} is negative, the differences between the perturbation solution and the solution to the complete problem are undetectable. The values of the parameters μ_2, χ_1, χ_2 and γ are chosen in the text to be representative of their upper limits on magnitude and have no particular physical significance as such.

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